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CHARACTERIZATION OF UNIGRAPHIC AND UNIDIGRAPHIC INTEGER-PAIR SEQUENCES

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Given a graph (digraph) G with edge (arc) set $E(G)=\{(u_1, v_1), (u_2, v_2), \ldots, (u_q, v_q)\}$, where $q=|E(G)|$, we can associate with it an integer-pair sequence $S_G=((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ where a_i, b_i are the degrees (indegrees) of u_i, v_i respectively. An integer-pair sequence S is said to be graphic (digraphic) if there exists a graph (digraph) G such that $S_G=S$. In this paper we characterize unigraphic and unidigraphic integer-pair sequences.

1. Introduction and definitions

All graphs (digraphs) considered here are finite, without isolated vertices and without loops or multiple edges (arcs). Given a graph (digraph) G we denote its vertex set by $V(G)$ and edge (arc) set by $E(G)$. The edge (arc) joining vertex u to vertex v is denoted by uv.

Let G be a graph (digraph) with $E(G)=\{u_1v_1, u_2v_2, \ldots, u_qv_q\}$ where $q=|E(G)|$. Then by the integer-pair sequence S_G of G we mean the sequence $((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ where a_i, b_i are the degrees (indegrees) of u_i, v_i respectively.

Let $S=((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ be a sequence of ordered pairs of positive (non-negative) integers. We say $S' \cong S$ if S' can be obtained from S by a permutation of its members, and $S' \triangleq S$ if $S' \cong S''$ where S'' is obtained from S by interchanging a_i and b_i in some of the members of S. Then a graph (digraph) G is said to be a realization of S if $S_G \triangleq S$ ($S_G \cong S$). If S has a graph (digraph) as a realization, then S is said to be graphic (digraphic). Further, if any two graph (digraph) realizations of S are isomorphic, then S is said to be unigraphic (unidigraphic).

Integer-pair sequences were first introduced by Hakimi and Patrinos in [3], where it was considered to extend the concepts and results of degree sequences. Characterizations of graphic (digraphic) integer-pair sequences were also obtained in the same paper. In this paper we characterize unigraphic (unidigraphic) integer-pair sequences, thus solving a problem posed in [3]. For degree sequences the corresponding problem of characterizing unigraphic degree sequences has been solved in the case of graphs by Koren in [6]; and in Das [2] in the case of digraphs. Further results on integer-pair sequences have been obtained in [1, 2, 7].

We now introduce some definitions and notations. For definitions not given here and notations not explained the reader is referred to [4].

Let G be a graph (digraph) and $A, B \subseteq V(G)$. Then $G[A, B]$ is defined by the following: $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{uv \in E(G): u \in A, v \in B\}$. $G[A, A]$ is denoted by $G[A]$ sometimes. If $x \in V(G)$, then the degree (out-degree and indegree respectively) of x in G is denoted by $d_G(x)(d_G^+(x)$ and $d_G^-(x)$ respectively). By $G \to I(xuyv) \to H$ we mean that we get H from G by replacing xu, yv by xv, yu in $E(G)$ where x, y, u, v are distinct vertices in $V(G)$ such that $xu, yv \in E(G)$; $yu, xv \notin E(G)$ and $d_G(u) = d_G(v)$ $(d_G^-(u) = d_G^-(v))$. Then, clearly, H is also a graph (digraph) and $S_H \cong S_G$.

Now let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ be a sequence of ordered pairs of positive integers.

We then have (following [3]): S_1 is the sequence $(a_1, b_1, a_2, b_2, \ldots, a_q, b_q)$; $S_2 = \{d_1, d_2, \ldots, d_n\}$ is the set of distinct members of S_1; $k'(r, s)$ is the number of times the ordered integer-pair (r, s) occurs in S;

$$k(r, s) = \begin{cases} k'(r, s) + k'(s, r), & \text{if } r \neq s, \\ k'(r, s), & \text{if } r = s; \end{cases}$$

$n(r)$ is the number of times r occurs in S_1;

$$l_i = \frac{n(d_i)}{d_i} \quad \text{for} \quad 1 \leq i \leq n.$$

Note that if S is graphic and G is a graph realization of S then there are l_i vertices of degree d_i in G (as shown in [3]) and hence l_i is a positive integer.

Hence for graphic S we have the following: For $1 \leq i \leq n$ we define:

$$X_i = (2k(d_i, d_i))(\text{mod } l_i) + \sum_{j \neq i} (k(d_i, d_j)(\text{mod } l_i)),$$

$$Y_i = (-2k(d_i, d_i))(\text{mod } l_i) + \sum_{j \neq i} ((-k(d_i, d_j))(\text{mod } l_i)).$$

Π_{ii} is the following sequence of length l_i: $(2k(d_i, d_i))(\text{mod } l_i)$ of the members are $\{2k(d_i, d_i)/l_i\}$ and the remaining, if any, are $[2k(d_i, d_i)/l_i]$. (As usual, $\{x\}$ denotes the least integer not less than x and $[x]$ denotes the greatest integer not greater than x.)

For $1 \leq i \neq j \leq n$ we define Π_{ij} to be the following pair of sequences $[(r_1^i, r_2^i, \ldots, r_{l_i}^i), (r_1^j, r_2^j, \ldots, r_{l_j}^j)]$ where $k(d_i, d_j)(\text{mod } l_m)$ of the r_p^m's are $\{k(d_i, d_j)/l_m\}$ and the remaining, if any, are $[k(d_i, d_j)/l_m]$ for $m = i, j$.

Also, when S is graphic, any graph realization G of S is taken to be on the vertex set $V = \bigcup_{i=1}^n V_i$ where $|V_i| = l_i$ and $d_G(x) = d_i$ for all $x \in V_i$, $1 \leq i \leq n$. $G[V_i, V_j]$ is denoted by G_{ij} and $G[V_i]$ by G_i or G_{ii}.

For the bipartite graph G_{ij}, $i \neq j$, the bipartition is always taken to be $V_i \cup V_j$ and $\Delta_{ij}(G)$, $\delta_{ij}(G)$ denote respectively the maximum, minimum degree in G_{ij} of a vertex in V_i. G_{ij} is said to be semiregular on both sides if

$$\Delta_{ij}(G)-\delta_{ij}(G)\leqslant 1 \quad \text{and} \quad \Delta_{ji}(G)-\delta_{ji}(G)\leqslant 1.$$

Given a pair of sequences $[\phi_1, \phi_2]$ we say it has a realization by bipartite graph if there is a bipartite graph G, with bipartition $V_1 \cup V_2$, such that the degrees in G of the vertices in V_m are given by ϕ_m for $m=1,2$. (Pairs of sequences with unique realization by bipartite graphs have been characterized by Koren in [5].) Then we also write $\Pi(G)=[\phi_1, \phi_2]$.

A graph G is said to be semiregular if $\Delta(G)-\delta(G)\leqslant 1$ where $\Delta(G)$, $\delta(G)$ denote respectively the maximum, minimum degree of a vertex in G.

$x \mid y$ means x divides y and $x \nmid y$ means x does not divide y.

The degree sequence of a graph G, written $\Pi(G)$, is the sequence of the degrees of the vertices of G. Two sequences Π_1, Π_2 are equal if Π_1 can be obtained by a permutation of the members of Π_2. Similarly $[\Pi_1, \Pi_2]=[\phi_1, \phi_2]$ if $\Pi_1=\phi_1$ and $\Pi_2=\phi_2$.

Further definitions, required only in the case of digraphs, will be given in the section on digraphs.

2. Unigraphic integer-pair sequences

We first give two structural results which will be used repeatedly in the proof of the necessity of the characterizing theorem.

Lemma 2.1. *Let G be a semiregular graph with q edges and n vertices. If $2\leqslant q\leqslant \frac{1}{2}n(n-1)-2$, then there exist distinct $x, y, v \in V(G)$ such that $xv \notin E(G)$, $yv \in E(G)$ and $d_G(x)\geqslant d_G(y)$.*

Proof. Let a be the maximum degree of a vertex in $V(G)$. If $a=1$, then there are four vertices of degree 1. Two of these, which are nonadjacent, may be taken to be x and y and v the only vertex adjacent to y. Similarly, if $a=n-1$, then there are four vertices of degree $n-2$ and two of these which are adjacent may be taken as x and y and v the only vertex non-adjacent to x.

So we suppose $2\leqslant a\leqslant n-2$. Let $x\in V(G)$ be such that $d_G(x)=a$. So there is $v\in V(G)$ such that $xv\notin E(G)$. As G is semiregular so $d_G(v)\geqslant 1$. Hence there is $y\in V(G)$ such that $yv\in E(G)$. This choice of x, y and v serves. Hence the lemma is proved.

Lemma 2.2. *Let G be a bipartite graph with bipartition $V_1 \cup V_2$, and with q edges, which is semiregular on both sides. If $2\leqslant q\leqslant mn-2$ where $|V_1|=n\geqslant 2$ and $|V_2|=m\geqslant 2$, then there exist $x, y\in V_1$ and $u, v\in V_2$ such that $xu, yv\in E(G)$ and $xv, yu\notin E(G)$.*

Proof. Let a_i be the maximum degree of a vertex in V_i, $i=1,2$. As before we can see that if $a_1=1$ or m ($a_2=1$ or n) the lemma is true. So we suppose that $2\leq a_1\leq m-1$ and $2\leq a_2\leq n-1$. Let $x\in V_1$ be such that $d_G(x)=a_1$. So there is $v\in V_2$ such that $xv\notin E(G)$. Now $d_G(v)\geq 1$ and so there is $y\in V_1$ such that $yv\in E(G)$. As $d_G(x)\geq d_G(y)$ so y is not joined to all the vertices that x is joined to. Hence there is $u\in V_2$ such that $yu\notin E(G)$, $xu\in E(G)$. This proves the lemma.

We will use the following result on unigraphic degree sequences:

Lemma 2.3 (Koren [6]). *Let $\Pi=a^n$ be a graphic degree sequence. Then Π is unigraphic if and only if $a\in\{1, n-2, n-1\}$ or $\Pi=2^5$. (a^n is the sequence of n a's.)*

We will use the following canonical realization of a graphic integer-pair sequence.

Lemma 2.4 (Rao and Taneja [7]). *If S is a graphic integer-pair sequence, there is a graph realization G of S such that for every i, j, $1\leq i\neq j\leq n$, G_i is semiregular and G_{ij} is semiregular on both sides.*

In what follows till the statement of the theorem in this section, we take S to be a unigraphic integer-pair sequence and G to be the canonical realization of Lemma 2.4. Note that for $1\leq i, j\leq n$ we have $\Pi(G_{ij})=\Pi_{ij}$. We also make the following notational simplifications: we write Δ_{ij} for $\Delta_{ij}(G)$, δ_{ij} for $\delta_{ij}(G)$, $d_{ij}(x)$ for $d_{G_{ij}}(x)$ and $d(x)$ for $d_G(x)$.

We now state and prove a series of assertions about unigraphic S. These will be required to prove the necessity in the characterizing theorem.

Assertion 1. *Either $X_i=Y_i=0$ or $X_i=l_i$ or $Y_i=l_i$ for $1\leq i\leq n$.*

Proof. For $l_i=1$ it is clearly true. So let $l_i\geq 2$ and suppose that the assertion does not hold. Now as S is graphic so $l_i\mid X_i$ and $l_i\mid Y_i$. Also $X_i=0$ if and only if $Y_i=0$. So we have that $X_i, Y_i\geq 2l_i$.

So there exist $x, y\in V_i$ and j, $1\leq j\leq n$, such that $d_{ij}(x)>d_{ij}(y)$. As $d(x)=d(y)=d_i$ so there is m, $1\leq m\leq n$, such that $d_{im}(y)>d_{im}(x)$. As $X_i, Y_i\geq 2l_i$ so each vertex of V_i has degree $\Delta_{ik}(\delta_{ik})$ in G_{ik}, where $\Delta_{ik}\neq\delta_{ik}$, for at least two different values of k, $1\leq k\leq n$. So let p, r be such that $d_{ip}(x)=\Delta_{ip}\neq\delta_{ip}$ and $d_{ir}(x)=\delta_{ir}\neq\Delta_{ir}$ and $\{p,r\}\cap\{j,m\}=\emptyset$.

Claim. *$d_{ip}(y)=\Delta_{ip}$ and $d_{ir}(y)=\delta_{ir}$.*

If not, then by interchanging the neighbourhoods of x and y in both G_{ij} and G_{im} we will get a realization H of S which has one vertex less than G with the property that it is joined to Δ_{ij} vertices in V_j, Δ_{ip} in V_p, δ_{im} in V_m and δ_{ir} in V_r.

This implies that S is not unigraphic. Hence the claim.

Now there is $z \in V_i$ such that $d_{ip}(z) = \delta_{ip}$.

Suppose $d_{im}(z) = \delta_{im}$. Then there are two cases. First if there exists $s \notin \{j, p, m, r\}$ such that $d_{is}(z) = \Delta_{is} \neq \delta_{is}$, $d_{is}(y) = \delta_{is}$, then interchanging neighbourhoods of y and z in G_{ip} and G_{is} we get H and then as above we can get a realization which is not isomorphic to H. So this case cannot occur and hence we have $d_{ij}(z) = \Delta_{ij}$ and $d_{ir}(z) = \Delta_{ir}$ as $X_i \geq 2l_i$. Interchanging neighbourhoods of y and z in G_{im} and G_{ij} we get a realization which has in V_i one more vertex, than G has, which is joined to Δ_{ir} points in V_r and δ_{ij} points in V_j.

So we have $d_{im}(z) = \Delta_{im}$. Then as before by comparing z with x we see that $d_{ij}(z) = \Delta_{ij}$ and $d_{ir}(z) = \delta_{ir}$. So there is $w \in V_i$ such that $d_{ir}(w) = \Delta_{ir}$. Hence there is $u_r \in V_r$ such that $wu_r \in E(G)$, $zu_r \notin E(G)$. Similarly there is t (t may be same as j or m) such that $d_{it}(w) = \delta_{it} < \Delta_{it} = d_{it}(z)$. Hence there is $u_t \in V_t$ such that $zu_t \in E(G)$, $wu_t \notin E(G)$. $G \to I(wu_r zu_t) \to G'$. Now in G' we interchange the neighbourhoods of z and y in both $G'[V_i, V_r]$ and $G'[V_i, V_p]$ to get realization H in which y is adjacent to δ_{ip} points in V_p and Δ_{ir} points in V_r, contradicting the claim.

Hence Assertion 1 is proved.

Note that $X_i = l_i$ ($Y_i = l_i$) implies that each vertex of V_i has degree $\Delta_{ij}(\delta_{ij})$ in some G_{ij}, where $\Delta_{ij} \neq \delta_{ij}$, exactly once. Also $X_i = Y_i = l_i$ implies that there are exactly two distinct values of j such that $\Delta_{ij} \neq \delta_{ij}$.

Assertion 2. *If $i \neq j$; $l_i \geq 2$ and $l_i \leq k(d_i, d_j) \leq l_i l_j - l_i$, then Π_{ij} has unique realization by bipartite graph; $X_i, Y_i \leq l_i$; for all $m \neq j, i$, with $l_m \geq 2$, either*

$$0 \leq k(d_i, d_m) \leq 1 \quad \text{or} \quad l_i l_m - 1 \leq k(d_i, d_m) \leq l_i l_m;$$

and either

$$0 \leq k(d_i, d_i) \leq 1 \quad \text{or} \quad \tfrac{1}{2} l_i (l_i - 1) - 1 \leq k(d_i, d_i) \leq \tfrac{1}{2} l_i (l_i - 1).$$

Further if $l_i \mid k(d_i, d_j)$, then for all $m \neq j, i$, $k(d_i, d_m) = l_i l_m$ or 0; and $k(d_i, d_i) = \frac{1}{2} l_i (l_i - 1)$ or 0.

Proof. Clearly Π_{ij} has unique realization by bipartite graph. We require the following claim to prove the rest of the assertion.

Claim. *Let $x, y \in V_i$, then there is a bipartite realization of Π_{ij} with bipartition $V_i \cup V_j$ such that there is a $w \in V_j$ which is joined to x but not to y.*

Let $V_j = \{v_1, \dots, v_{l_j}\}$ where $d_{ji}(v_k) \geq d_{ji}(v_{k+1})$. Then y can be joined to v_1 to $v_{d_{ij}(y)}$ and x to $v_{d_{ij}(y)+1} = w$ in G_{ij} as $d_{ji}(v_1) - d_{ji}(w) \leq 1$ and $d_{ij}(y) \leq \Delta_{ij} \leq l_j - 1$.

Now suppose there is an $m \neq j, i$ such that $l_m \geq 2$ and $2 \leq k(d_i, d_m) \leq l_i l_m - 2$. Then G_{im} satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i$, $v \in V_m$

such that $d_{im}(x) \geqslant d_{im}(y)$ and $xv \notin E(G_{im})$, $yv \in E(G_{im})$. Also by Claim there is $w \in V_j$ such that $xw \in E(G)$, $yw \notin E(G)$. $G \to I(xwyv) \to H$. Then $\Pi(H[V_i, V_m]) \neq \Pi_{im}$ as $\Delta_{im}(H) - \delta_{im}(H) \geqslant 2$. Contradiction to the fact that S is unigraphic.

Similarly we get a contradiction using Lemma 2.1 if

$$2 \leqslant k(d_i, d_i) \leqslant \tfrac{1}{2} l_i(l_i - 1) - 2.$$

Now let $l_i \mid k(d_i, d_j)$. Then for all $x, y \in V_i$, $d_{ij}(x) = d_{ij}(y)$. Suppose there is $v \in (V(G) - V_j)$ such that $xv \notin E(G)$, $yv \in E(G)$ for some $x, y \in V_i$. Then we get a w as in Claim. $G \to I(xwyv) \to H$ and $\Pi(H[V_i, V_j]) \neq \Pi_{ij}$. Contradiction. Hence there is no such v. Hence the second paragraph of Assertion 2 holds.

Finally to show that $X_i, Y_i \leqslant l_i$. Suppose not. Suppose $X_i \geqslant 2l_i$. Then by the second paragraph of Assertion 2, just proved, we get that $l_i \nmid k(d_i, d_j)$. So let $x \in V_i$ be such that $d_{ij}(x) = \Delta_{ij} \neq \delta_{ij}$. As $X_i \geqslant 2l_i$ so there is distinct r such that $d_{ir}(x) = \Delta_{ir} \neq \delta_{ir}$. Let $y \in V_i$ be such that $d_{ir}(y) = \delta_{ir}$. Then there is $v \in V_r$ such that $xv \in E(G)$, $yv \notin E(G)$. By Claim there is a $w \in V_j$ such that $xw \notin E(G)$, $yw \in E(G)$. $G \to I(xvyw) \to H$. Then $\Pi(H[V_i, V_j]) \neq \Pi_{ij}$. Hence $X_i \leqslant l_i$. Similarly $Y_i \leqslant l_i$.

Hence Assertion 2 is proved.

Assertion 3. *If $i \neq j$, $l_i, l_j \geqslant 2$ and $0 < k(d_i, d_j) < l_i$, then $X_i = l_i$.*

Proof. Suppose not. Then $X_i \geqslant 2l_i$ and by Assertion 1 $Y_i = l_i$. If $k(d_i, d_j) = 1$, then all but one vertex of V_i have degree δ_{ij} in G_{ij} where $\delta_{ij} \neq \Delta_{ij}$. Hence there is just one more k such that $\Delta_{ik} \neq \delta_{ik}$, and only the remaining vertex of V_i has degree δ_{ik} in G_{ik}. Then we get that $X_i = l_i$ also as the only non-zero contribution to X_i are from $k(d_i, d_j)$ and $k(d_i, d_k)$.

So $k(d_i, d_j) \geqslant 2$. Hence G_{ij} satisfies conditions of Lemma 2.2 and so we can get $x, y \in V_i$ and $u, v \in V_j$ such that $xu, yv \in E(G)$ and $xv, yu \notin E(G)$. Also as $\delta_{ij} = 0$ so $d_{ij}(x) = d_{ij}(y) = \Delta_{ij} = 1$.

As $X_i \geqslant 2l_i$ so there is $m \neq j$ such that $d_{im}(x) = \Delta_{im} \neq \delta_{im}$. Hence there exist $z \in V_i$, $w \in V_m$ such that $xw \in E(G)$, $zw \notin E(G)$ and $d_{im}(z) = \delta_{im}$. If $yw \notin E(G)$, then $G \to I(xwyv) \to H$. Then in H, x has degree 2 in $H[V_i, V_j]$ but there is no vertex of V_i of degree 2 in G_{ij} implying S is not unigraphic. Hence $yw \in E(G)$.

As $Y_i = l_i$ so $d_{ij}(z) = \Delta_{ij} = 1$. Let $p \in V_j$ such that $zp \in E(G)$. So at least one of x, y is nonadjacent to p. Let $xp \notin E(G)$. $G \to I(xwzp) \to H$. Then as above we get a contradiction.

Hence Assertion 3 is proved.

Assertion 4. *If $i \neq j$; $l_i, l_j \geqslant 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j$, then $Y_i = l_i$.*

Proof. Let $\bar{G}$ be the complement of G. Then $S_{\bar{G}}$ is unigraphic. So Assertion 3 holds for $S_{\bar{G}}$. This implies that Assertion 4 holds for S.

Assertion 5. *If $i \neq j$; $l_i, l_j \geq 2$ and $1 < k(d_i, d_j) < l_i$, then for all $m \neq j, i$ either $k(d_i, d_m) = l_i l_m$ or $0 \leq k(d_i, d_m) < l_i$; and either*

$$k(d_i, d_i) < \tfrac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geq \tfrac{1}{2} l_i (l_i - 1) - 1.$$

Proof. Now by Lemma 2.2 there exist $x, y \in V_i$ and $u, v \in V_j$ such that $xu, yv \in E(G)$ and $xv, yu \notin E(G)$. Hence $d_{ij}(x) = d_{ij}(y) = \Delta_{ij} = 1$. Also applying Assertion 3 to the hypothesis we get that $X_i = l_i$.

Now suppose there is $m \neq j, i$ such that $l_i \leq k(d_i, d_m) < l_i l_m$. Then clearly $l_m \geq 2$. Hence using Assertion 2 we get that $k(d_i, d_m) > l_i l_m - l_i$ or $k(d_i, d_m) < l_i$. So $l_i l_m - l_i < k(d_i, d_m) < l_i l_m$. As $X_i = l_i$ so $d_{im}(x) = d_{im}(y) = \delta_{im} = l_m - 1 \geq 1$. Hence without loss of generality we can take G_{im} (constructed as in proof of Claim in Assertion 2) to be so that there is $w \in V_m$ such that $yw \in E(G)$, $xw \notin E(G)$. $G \to I(xuyw) \to H$ and as before H is not isomorphic to G. Contradiction.

Now if

$$\tfrac{1}{2} l_i \leq k(d_i, d_i) \leq \tfrac{1}{2} l_i (l_i - 1) - 2.$$

Then $l_i \geq 4$. Then also $d_{ii}(x) = d_{ii}(y) = \delta_{ii}$ where $1 \leq \delta_{ii} \leq l_i - 2$. ($\delta_{ii}$ may be equal to Δ_{ii}). Now we let $V_i = \{v_i, v_2, \ldots, v_{l_i-2}, x = v_{l_i-1}, y = v_{l_i}\}$ such that $d_{ii}(v_k) \geq d_{ii}(v_{k+1})$ for $k = 1, 2, \ldots, l_i - 1$. If $\delta_{ii} \leq l_i - 3$, then G_{ii} is constructed as follows: x is joined to v_1 to $v_{d_{ii}(x)}$. Then y is joined to $v_{d_{ii}(x)+1} = w$. If $\delta_{ii} = l_i - 2$, then there are at least 4 vertices of degree $l_i - 2$ as $k(d_i, d_i) \leq \tfrac{1}{2} l_i (l_i - 1) - 2$; and we take G_{ii} such that $xy \in E(G_{ii})$. So there is $w \in V_i$ such that $w \neq x$ and $xw \in E(G_{ii})$. So in any case we can get a G_{ii} such that there is a $w \in V_i$, $w \neq x$ and $yw \in E(G_{ii})$, $xw \notin E(G_{ii})$. $G \to I(xuyw) \to H$ and $iI(H[V_i]) \neq \Pi_{ii}$. A contradiction.

Hence Assertion 5 is proved.

Assertion 6. *If $i \neq j$, $l_i, l_j \geq 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j - 1$, then for all $m \neq j, i$ either $k(d_i, d_m) = 0$ or $l_i l_m - l_i < k(d_i, d_m) \leq l_i l_m$; and either*

$$k(d_i, d_i) > \tfrac{1}{2}(l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leq 1.$$

Proof. Apply Assertion 5 to $S_{\bar{G}}$.

Assertion 7. *If i, j, r are all distinct, $X_i = l_i$; $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_r) < l_i l_r$, then for all $m \neq i, j, r$ either $k(d_i, d_m) = l_i l_m$ or $0 \leq k(d_i, d_m) < l_i$; and either*

$$k(d_i, d_i) < \tfrac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geq \tfrac{1}{2} l_i (l_i - 1) - 1.$$

Proof. Note $X_i = l_i$ implies $l_i \geq 2$. Also if Assertion 2 could be applied to either i, j or i, r then we are done. Hence, as Assertion 2 cannot be applied, we can get $x, y \in V_i$, $u \in V_j$, $v \in V_r$ such that $xu, yv \in E(G)$ and $xv, yu \notin E(G)$. The rest of the proof is similar to that of Assertion 5.

Assertion 8. *If i, j, r are all distinct; $Y_i = l_i$; $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_r) < l_i l_r$,* *then for all $m \neq i, j, r$ either $k(d_i, d_m) = 0$ or $l_i l_m - l_i < k(d_i, d_m) \leqslant l_i l_m$; and either*

$$k(d_i, d_i) > \tfrac{1}{2} l_i (l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leqslant 1.$$

Proof. Apply Assertion 7 to $S_{\bar{G}}$.

Assertion 9. *If $k(d_i, d_i) \neq 0$ or $\frac{1}{2} l_i (l_i - 1)$ and $l_i \mid 2k(d_i, d_i)$, then either*

$$k(d_i, d_i) = \tfrac{1}{2} l_i \quad \text{or} \quad \tfrac{1}{2} l_i (l_i - 2) \quad \text{or} \quad k(d_i, d_i) = l_i = 5;$$

and for all $m \neq i$, $k(d_i, d_m) = l_i l_m$ or 0.

Proof. Since Π_{ii} is unigraphic so the values of $k(d_i, d_i)$ are obtained using Lemma 2.3. If for some $m \neq i$ we have $0 < k(d_i, d_m) < l_i l_m$, then we can get $x, y \in V_i$, $w \in V_m$ such that $xw \in E(G)$, $yw \notin E(G)$ as G_{im} is semiregular on both sides. Also, whichever of the permitted values $k(d_i, d_i)$ takes, we can get a G_{ii} such that there is $u \in V_i$, $u \neq x$ and $ux \notin E(G_{ii})$, $uy \in E(G_{ii})$. $G \to I(xwyu) \to H$ and $\Pi(H[V_i]) \neq \Pi_{ii}$. A contradiction. Hence Assertion 9 is proved.

Assertion 10. *If $l_i \nmid 2k(d_i, d_i)$, then Π_{ii} is unigraphic.*

Proof. The proof is immediate.

Now we state and prove the theorem characterizing unigraphic integer-pair sequences.

Theorem 2.1. *An integer-pair sequence S is unigraphic if and only if S is graphic and the following conditions are satisfied.*

C1. *Either $X_i = Y_i = 0$ or $X_i = l_i$ or $Y_i = l_i$ for $1 \leqslant i \leqslant n$.*

C2. *If $i \neq j$; $l_i \geqslant 2$ and $l_i \leqslant k(d_i, d_j) \leqslant l_i l_j - l_i$, then Π_{ij} has unique realization by bipartite graph; $X_i, Y_i \leqslant l_i$; for all $m \neq j, i$, with $l_m \geqslant 2$, either $0 \leqslant k(d_i, d_m) \leqslant 1$ or $l_i l_m - 1 \leqslant k(d_i, d_m) \leqslant l_i l_m$; and either*

$$0 \leqslant k(d_i, d_i) \leqslant 1 \quad \text{or} \quad \tfrac{1}{2} l_i (l_i - 1) - 1 \leqslant k(d_i, d_i) \leqslant \tfrac{1}{2} l_i (l_i - 1).$$

Further, if $l_i \mid k(d_i, d_j)$, then for all $m \neq j, i$, $k(d_i, d_m) = l_i l_m$ or 0; and $k(d_i, d_i) = \frac{1}{2} l_i (l_i - 1)$ or 0.

C3. *If $i \neq j$; $l_i, l_j \geqslant 2$ and $0 < k(d_i, d_j) < l_i$, then $X_i = l_i$.*

C4. *If $i \neq j$; $l_i, l_j \geqslant 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j$, then $Y_i = l_i$.*

C5. *If $i \neq j$; $l_i, l_j \geqslant 2$ and $1 < k(d_i, d_j) < l_i$, then for all $m \neq j, i$ either $k(d_i, d_m) = l_i l_m$ or $0 \leqslant k(d_i, d_m) < l_i$; and either*

$$k(d_i, d_i) < \tfrac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geqslant \tfrac{1}{2} l_i (l_i - 1) - 1.$$

C6. *If $i \neq j$; $l_i, l_j \geqslant 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j - 1$, then for all $m \neq j, i$ either $k(d_i, d_m) = 0$ or $l_i l_m - l_i < k(d_i, d_m) \leqslant l_i l_m$; and either*

$$k(d_i, d_i) > \tfrac{1}{2} l_i (l_i - 2) \text{ or } k(d_i, d_i) \leqslant 1.$$

C7. *If i, j, r are all distinct; $X_i = l_i$; $0 < k(d_i, d_j) < l_il_j$ and $0 < k(d_i, d_r) < l_il_r$, then for all $m \neq i, j, r$ either $k(d_i, d_m) = l_il_m$ or $0 \leqslant k(d_i, d_m) < l_i$; and either*

$$k(d_i, d_i) < \tfrac{1}{2}l_i \quad \text{or} \quad k(d_i, d_i) \geqslant \tfrac{1}{2}l_i(l_i - 1) - 1.$$

C8. *If i, j, r are all distinct; $Y_i = l_i$; $0 < k(d_i, d_j) < l_il_j$ and $0 < k(d_i, d_r) < l_il_r$, then for all $m \neq i, j, r$ either $k(d_i, d_m) = 0$ or $l_il_m - l_i < k(d_i, d_m) \leqslant l_il_m$; and either*

$$k(d_i, d_i) > \tfrac{1}{2}l_i(l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leqslant 1.$$

C9. *If $k(d_i, d_i) \neq 0$ or $\frac{1}{2}l_i(l_i - 1)$ and $l_i \mid 2k(d_i, d_i)$, then either*

$$k(d_i, d_i) = \tfrac{1}{2}l_i \quad \text{or} \quad \tfrac{1}{2}l_i(l_i - 2) \quad \text{or} \quad k(d_i, d_i) = l_i = 5;$$

and for all $m \neq i$, $k(d_i, d_m) = l_il_m$ or 0.

C10. *If $l_i \nmid 2k(d_i, d_i)$, then Π_{ii} is unigraphic.*

Proof. The necessity of conditions C1 to C10 follow from Assertions 1 to 10 respectively.

Sufficiency. Let S be an integer-pair sequence which is graphic and satisfies the conditions C1 to C10. Let H be a realization of S and let G be the canonical realization of S obtained from Lemma 2.4. Analogous to the notations developed for G, we have the following for H: H_{ij} denotes $H[V_i, V_j]$, $d'_{ij}(x)$ denotes the degree of x in H_{ij}.

We will prove the sufficiency by showing that H is isomorphic to G. To this end we state and prove the following claims.

Claim 1. *For fixed i and all j the degree sequence of the vertices of V_i in the graph H_{ij} is same as in G_{ij}.*

For $l_i = 1$ it is clearly true. So let $l_i \geqslant 2$. Also then we need only check for $j \neq i$, such that $l_j \geqslant 2$ and $0 < k(d_i, d_j) < l_il_j$.

Suppose for all $j \neq i$, $k(d_i, d_j) = 0$ or l_il_j, then H_{ii} (respectively G_{ii}) has to be regular as the degree in H (G) of all vertices of V_i is same. Thus $\Pi(H_{ii}) = \Pi_{ii}$ and the claim holds.

So now we suppose there exists $j \neq i$ such that $0 < k(d_i, d_j) < l_il_j$. Then we have the following five exhaustive cases.

Case (a). There exists $j \neq i$ such that $l_j \geqslant 2$ and $l_i \leqslant k(d_i, d_j) \leqslant l_il_j - l_i$. Now if $l_i \mid k(d_i, d_j)$, then from C2 we see that all vertices of V_i have same degree in H_{ij} (G_{ij}). Hence the claim holds.

So let $l_i \nmid k(d_i, d_j)$. So $X_i \neq 0$, $Y_i \neq 0$. Hence $X_i = Y_i = l_i$ from C2. This implies that there are exactly two values of r such that $\Delta_{ir} \neq \delta_{ir}$. One of them is j, let the other be m. Then, from C2, we get that for all $r \neq j, m, i$ $k(d_i, d_r) = 0$ or l_il_r and so for all these r claim holds. Also, from C2, either

$$0 < k(d_i, d_i) \leqslant 1 \quad \text{or} \quad \tfrac{1}{2}l_i(l_i - 1) - 1 \leqslant k(d_i, d_i) \leqslant \tfrac{1}{2}l_i(l_i - 1)$$

and so the claim is true for i too. If $m \neq i$, then, from C2, H_{ii} (G_{ii}) has exactly one edge or nonedge, if $l_m \geq 2$; and if $l_m = 1$, then H_{im} and G_{im} are isomorphic. Hence in any case claim is true for m. As $\Delta_{ir} \neq \delta_{ir}$ only for $r = m, j$ and claim is true for m so it is true too for j (as all vertices of V_i have same degree in H). Hence the claim holds.

Case (b). Not in Case (a) and there exists $j \neq i$ such that $l_j \geq 2$ and $1 < k(d_i, d_j) < l_i$.

From C3 we get that $X_i = l_i$. If $k(d_i, d_i) < \frac{1}{2}l_i$, then, from C5, we get that $X_i = \sum_{j \in J} k(d_i, d_j) + 2k(d_i, d_i)$ where for all $j \in J$ we have $l_i \nmid k(d_i, d_j)$ and $i \notin J$. Also, from C5, we get that for all $m \notin J$, $m \neq i$ $k(d_i, d_m) = 0$ or $l_i l_m$. Hence each vertex of V_i has exactly one of the edges counted in the expression for X_i above as $X_i = l_i$. Thus the claim holds.

If $k(d_i, d_i) = \frac{1}{2}l_i(l_i - 1) - 1$. Then $\Pi(H_{ii}) = \Pi_{ii}$. Then as $X_i = l_i$ and $2 \leq k(d_i, d_j) < l_i$ so for all $m \neq j, i$ $\Delta_{im} = \delta_{im}$. As we are not in Case (a) so $k(d_i, d_m) = 0$ or $l_i l_m$. Moreover $k(d_i, d_j) = 2$. Hence the claim holds.

If $k(d_i, d_i) = \frac{1}{2}l_i(l_i - 1)$, then, from C5, we get that $X_i = \sum_{j \in J} k(d_i, d_j)$ with J defined as before. Similarly here, too, the claim holds.

Case (c). Not in Case (a) and there exists $j \neq i$ such that $l_j \geq 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j - 1$.

C6 ensures that this follows by considering $\bar{H}$ and Case (b).

Case (d). Not in any of the previous cases and there exist $j, m \neq i$ such that $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_m) < l_i l_m$.

If $X_i = Y_i = 0$, then we are in Case (a). Hence by C1 we have either $X_i = l_i$ or $Y_i = l_i$. Suppose $X_i = l_i$. Then we use C7 and proof is similar to that of Case (b). If $Y_i = l_i$, then C8 shows that we are in a case similar to Case (c).

Case (e). Not in any of the previous cases. Hence there exists $j \neq i$ such that either $k(d_i, d_j) = 1$ or $l_i l_j - 1$; or $l_j = 1$ with $0 < k(d_i, d_j) < l_i l_j$; and for all $m \neq i, j$ $k(d_i, d_m) = 0$ or $l_i l_m$. Then clearly claim is true for j and hence also $\Pi(H_{ii}) = \Pi_{ii}$.

Thus the claim is proved.

From Claim 1 we immediately have the following.

Claim 2. *$\Pi(H_{ij}) = \Pi_{ij}$ for $1 \leq i, j \leq n$.*

So now we may denote $\Delta_{ij}(H)$ ($\delta_{ij}(H)$) by Δ_{ij} (δ_{ij}) also. We now require the following definitions for the next two claims.

We say that i is paired if there is a j such that

$$l_i, l_j \geq 2 \quad \text{and} \quad \max\{l_i, l_j\} \leq k(d_i, d_j) \leq l_i l_j - \max\{l_i, l_j\};$$

further, we say that i, j are a pair and i is paired with j. Otherwise we say that i is not paired.

For $x \in V_i$ we define $f(x) = (d_{i1}(x), d_{i2}(x), \ldots, d_{in}(x))$, $f'(x) = (d'_{i1}(x), d'_{i2}(x), \ldots, d'_{in}(x))$.

A map $\phi: V \to V$ is said to be permissible if ϕ is 1–1 and for $1 \leq i \leq n$, $x \in V_i$ implies $\phi(x) \in V_i$ and $f'(x) = f(\phi(x))$.

Observe that Claim 2 and C1 shows that a permissible map exists.

If uv is an edge (nonedge) in H, then we say ϕ maps it onto an edge (nonedge) if $\phi(u)\phi(v)$ is an edge (nonedge) in G. Hence a permissible map ϕ which maps all edges of H onto edges will be an isomorphism of H onto G. Note that if $l_i = 1$, then any permissible map ϕ is an isomorphism from $H[V_i, V]$ onto $G[V_i, V]$. Hence we only consider i such that $l_i \geq 2$ to see how we may obtain an isomorphism from a given permissible map ϕ. To this end we have the following two claims.

Claim 3. *If i is paired with j and ϕ is any permissible map, then we can get a permissible map ϕ' such that $\phi' = \phi$ on $V-(V_i \cup V_j)$ and ϕ' maps all edges in H with at least one point in $V_i \cup V_j$ onto edges of G.*

By Claim 2 and C2 we get that $\Pi(H_{ij}) = \Pi_{ij}$ has unique realization by bipartite graph. So there is an isomorphism ψ from H_{ij} onto G_{ij} with $\psi(V_i) = V_i$. Define ϕ' as follows: $\phi' = \psi$ on $V_i \cup V_j$ and $\phi' = \phi$ on $V-(V_i \cup V_j)$. To check that this ϕ' serves. Clearly all edges of H_{ij} are mapped onto edges by ϕ'.

If $l_i \mid k(d_i, d_j)$, then, from C2, we see that for all $m \neq i, j$ all edges of H_{im} are mapped onto edges. Also for $x \in V_i$: $f'(x) = f(\phi'(x))$. Similarly if $l_j \mid k(d_i, d_j)$.

If $l_i \nmid k(d_i, d_j)$, then $X_i, Y_i \neq 0$. So from C2 we get that $X_i = Y_i = l_i$ and hence there is exactly one $m \neq j$ such that $\Delta_{im} \neq \delta_{im}$. If $m \neq i$ and $l_m \geq 2$, then there is exactly one edge or nonedge in H_{im} by C2. The vertex x in V_i which has the only edge or nonedge in H_{im} has its degree in H_{ij} different from the degree in H_{ij} of all other vertices of V_i. Hence x is mapped by ψ, and hence ϕ', to y the vertex of same degree in G_{ij}. As a permissible map exists so $f'(x) = f(y)$. Hence in G also y has the only edge or nonedge in G_{im}.

If $m = i$, then there is exactly one edge or nonedge, say xy, in H_{ii} by C2. Then $d'_{ij}(x) = d'_{ij}(y)$ and for all $z \in V_i$ $z \neq x, y$ we have $d'_{ij}(z) \neq d'_{ij}(x)$. So xy is mapped by ϕ' onto the two vertices of V_i which have their degree in G_{ij} same as $d'_{ij}(x)$. Hence ϕ' maps xy onto the only edge or nonedge in G_{ij}. Similarly if $l_j \nmid k(d_i, d_j)$.

Clearly ϕ' is permissible in each case. Hence the claim is proved.

Claim 4. *If i is not paired; there is no $j \neq i$ such that $l_i \leq k(d_i, d_j) \leq l_i l_j - l_i$; and ϕ is any permissible map, then we can get a permissible map ϕ' such that $\phi' = \phi$ on $V - V_i$ and ϕ' maps all edges in H with at least one point in V_i onto edges.*

Suppose for all j H_{ij} is complete or empty, then clearly so is G_{ij} and hence we can take $\phi' = \phi$. So let there exist a j such that H_{ij} is neither complete nor empty. If $j = i$ and $l_i \mid 2k(d_i, d_i)$, then, by C9, we know that for all $m \neq i$, H_{im} is either complete or empty. Also by C9 and Lemma 2.3 we see that Π_{ii} is unigraphic. Hence there is an isomorphism ψ from H_{ii} onto G_{ii}. We then define ϕ' as follows: $\phi' = \psi$ on V_i and $\phi' = \phi$ on $V - V_i$. This ϕ' serves.

If $j=i$ and $l_i \nmid 2k(d_i, d_j)$, then there is $m \neq i$ such that $\Delta_{im} \neq \delta_{im}$. In particular G_{im} is neither complete nor empty. So the only case remaining is that there is a $j \neq i$ such that $0 < k(d_i, d_j) < l_i l_j$. We make the following subcases, which are exhaustive.

Subcase (a). There exists $j \neq i$ such that $l_j \geq 2$ and $1 < k(d_i, d_j) < l_i$.

Then by C3 we know that $X_i = l_i$. Also, by C5, either

$$k(d_i, d_i) < \tfrac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geq \tfrac{1}{2} l_i (l_i - 1) - 1.$$

Suppose $k(d_i, d_i) = \frac{1}{2} l_i (l_i - 1) - 1$ and $k(d_i, d_j) \neq 0$. Applying Lemma 2.2 to H_{ij} we can get $x, y \in V_j$, $u, v \in V_i$ such that $xu, yv \in E(H)$. Note $u \neq v$ as $d'_{ij}(u) = d'_{ij}(v) = 1$. As ϕ is permissible so we can get $u', v' \in V_i$ such that $\phi(x)u'$, $\phi(y)v' \in E(G)$. Similarly $u' \neq v'$. Define ϕ' as follows: $\phi'(u) = u'$; $\phi'(v) = v'$; $\phi' = \phi$ on $V - V_i$; and ϕ' on $V_i - \{u, v\}$ is any 1–1 map onto $V_i - \{u', v'\}$. To check that ϕ' serves. As u, v (u', v') have degree $\Delta_{ij} = 1 \neq \delta_{ij}$ in H_{ij} (G_{ij}) and $X_i = l_i$ so they have degree $\delta_{ii} = l_i - 2$ in H_{ii} (G_{ii}). Hence the only nonedge in H_{ii} (G_{ii}) is uv $(u'v')$. So all edges of H_{ii} are mapped onto edges. The vertices of $V_i - \{u, v\}$ $(V_i - \{u', v'\})$ have degree $\Delta_{ii} \neq \delta_{ii}$. Hence, as $X_i = l_i$, for all $m \neq j, i$ all vertices of V_i have degree $\delta_{im} = 0$ or l_m, by C5, in H_{im}. Thus all edges of H_{ii}, H_{ij}, H_{im} for $m \neq j, i$ are mapped onto edges.

Suppose $k(d_i, d_i) < \frac{1}{2} l_i$ or $k(d_i, d_i) = \frac{1}{2} l_i (l_i - 1)$. Then from C5 it can be seen that $\Delta_{im} \neq \delta_{im}$ implies $\Delta_{im} = 1$; for $m \neq i$, $\Delta_{im} = \delta_{im}$ implies $\delta_{im} = 0$ or l_m; and $\Delta_{ii} = \delta_{ii}$ implies $\delta_{ii} = l_i - 1$ or 0. Let $m \neq i$, $\Delta_{im} = 1 \neq \delta_{im}$ and $x \in V_m$. Now as ϕ is permissible so $d'_{mi}(x) = d_{mi}(\phi(x)) = k$ (say). Let $xu_1, \ldots, xu_k \in E(H_{im})$ and $\phi(x)v_1, \ldots, \phi(x)v_k \in E(G_{im})$. Then we define $\phi'(u_r) = v_r$, $1 \leq r \leq k$. Further if $0 < k(d_i, d_i) < \frac{1}{2} l_i$ and $\{u_1 v_1, \ldots, u_k v_k\} = E(H_{ii})$ and $\{u'_1 v'_1, \ldots, u'_k v'_k\} = E(G_{ii})$ we define $\phi'(u_r) = u'_r$ and $\phi'(v_r) = v'_r$ for $1 \leq r \leq k$. Thus we can define ϕ' on V_i. ϕ' is well defined as $X_i = l_i$. Clearly this ϕ' serves.

Subcase (b). There exists $j \neq i$ such that $l_j \geq 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j - 1$.

C6 ensures that this follows from considering $\bar{H}$ and Subcase (a).

Subcase (c). Not in any of the previous subcases and there exist $j, m \neq i$ such that $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_m) < l_i l_m$.

If $X_i = Y_i = 0$, then we have $l_i \leq k(d_i, d_j) \leq l_i l_j - l_i$, a contradiction. Hence, by C1, either $X_i = l_i$ or $Y_i = l_i$. Suppose $X_i = l_i$ but $Y_i \neq l_i$. Then we have also for $r = j, m$ $0 < k(d_i, d_r) < l_i$ as we are not in any of the previous cases. Using this and C7 we proceed as in Subcase (a). Similarly we are done if $Y_i = l_i$ and $X_i \neq l_i$. If $X_i = Y_i = l_i$, then j, m are the only two values of r for which H_{ir} is not complete or empty. Then we can take $\phi' = \phi$.

Subcase (d). Not in any of the previous cases.

Here (same case as Case (e) of Claim 1) we take $\phi' = \phi$ if $l_i \mid 2k(d_i, d_i)$. If $l_i \nmid 2k(d_i, d_i)$, then by C10 we know that Π_{ii} is unigraphic. Let ψ be an isomorphism from H_{ii} onto G_{ii}. Then we define $\phi' = \phi$ on $V - V_i$ and $\phi' = \psi$ on V_i.

This proves the claim.

Now note that if i is paired with j, then i is not paired with any $m \neq j$ by C2.

Hence distinct pairs are disjoint. So we begin with a permissible map, say ϕ, which we know exists, and modify it successively for each distinct pair, according to Claim 3, and then for each i satisfying the conditions of Claim 4, according to Claim 4. Let ϕ_0 be the permissible map we get finally. Now, if any i is not covered in the above steps, then we know that it is not paired and there is a j such that $l_i, l_j \geq 2$ and $l_i \leq k(d_i, d_j) \leq l_i l_j - l_i$. Then as j is not paired with i and C2 is satisfied, there cannot be any $m \neq j$ such that $l_m \geq 2$ and $l_j \leq k(d_i, d_m) \leq l_j l_m - l_j$. Hence i has been covered in the above steps and so all edges of H_{ij} are mapped onto edges by ϕ_0. All other edges in H with at least one point in V_i are mapped onto edges by any permissible map: this follows from C2 and fact that any edge with at least one point in V_r, where $l_r = 1$, is mapped onto an edge by any permissible map. Hence we see that ϕ_0 is an isomorphism of H onto G.

This completes the proof of the theorem.

3. Unidigraphic integer-pair sequences

As there may be non-isolated vertices in a digraph with indegree 0 so we require the following analogous definitions in the case of digraphs.

Let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ be a sequence of ordered pairs of non-negative integers.

We then have (following [3]): A is the sequence $(a_1, a_2, \ldots, a_q)$; B is the sequence $(b_1, b_2, \ldots, b_q)$; A^* is the set of distinct members of A; $B^* = \{d_1, d_2, \ldots, d_n\}$ is the set of distinct members of B; $k'(r, s)$ is the number of times the ordered integer-pair (r, s) occurs in S; $k'(r)$ is the number of times r occurs in B; $k(r)$ is the number of times r occurs in A and B.

It is shown in [3] that if S is digraphic, then $0 \notin B^*$ and any digraph realization of S has $k'(d_i)/d_i$ vertices of indegree d_i. Hence $k'(d_i)/d_i$ is a positive integer. Also then $k(0)$ is the number of times 0 occurs in A.

Thus for digraphic S we have the following: For $1 \leq i \leq n$ we define:

$$l_i' = \frac{k'(d_i)}{d_i},$$

$$X_i' = \sum_{j=1}^{n} ((k'(d_j, d_i))(\text{mod } l_i')) + (k'(0, d_i)(\text{mod } l_i')),$$

$$Y_i' = \sum_{j=1}^{n} ((-k'(d_j, d_i))(\text{mod } l_i')) + ((-k'(0, d_i))(\text{mod } l_i')).$$

Also when S is digraphic any digraph realization G of S is considered on the vertex set $V(G) = \bigcup_{i=0}^{n} V_i$ where for $1 \leq i \leq n$, $|V_i| = l_i'$ and $d_G^-(x) = d_i$ for $x \in V_i$; if $0 \notin A^*$, then $V_0 = \emptyset$; and if $0 \in A^*$, then for $x \in V_0$, $d_G^-(x) = 0$.

As before for $0 \leq i, j \leq n$ we denote $G[V_i, V_j]$ by G_{ij} and $G[V_i]$ by G_i or G_{ii}. $\Delta_{ij}^+(G)$ and $\delta_{ij}^+(G)$ ($\Delta_{ij}^-(G)$ and $\delta_{ij}^-(G)$) denote respectively the maximum and minimum out-degree (indegree) in G_{ij} of a vertex in V_i (V_j).

For the bipartite digraph G_{ij} $(i \neq j)$ the bipartition is always taken to be $V_i \cup V_j$ and G_{ij} is said to be semiregular on both sides if G_{ij} considered as an undirected bipartite graph is. (Note the digraph G_{ij} is antisymmetric.)

The maximum and minimum outdegree (indegree) of a vertex in a digraph G is denoted respectively by $\Delta^+(G)$ and $\delta^+(G)$ ($\Delta^-(G)$ and $\delta^-(G)$). A digraph G is said to be semiregular if $\Delta^+(G)-\delta^+(G) \leq 1$ and $\Delta^-(G)-\delta^-(G) \leq 1$.

We now give a canonical realization of a digraphic integer-pair sequence.

Lemma 3.1. *If S is a digraphic integer-pair sequence, then there is a digraph realization G of S such that for every i, j, $1 \leq i \neq j \leq n$, G_i is semiregular and G_{ij} is semiregular on both sides.*

Proof. Let H be a digraph realization of S. Let M_{ij}^+ (M_{ij}^-) be the number of vertices in V_i (V_j) that have maximum outdegree (indegree) in $H[V_i, V_j]$.

Let

$$f(H)=\sum_{i=1}^{n}\sum_{j=1}^{n}(\Delta_{ij}^+(H)-\delta_{ij}^+(H)+\Delta_{ij}^-(H)-\delta_{ij}^-(H)+M_{ij}^+ + M_{ij}^-).$$

Then of all the realizations of S we choose one, G, such that $f(G)$ is minimum.

Claim. *G satisfies the conditions of Lemma* 3.1.

Suppose there is i such that G_i is not semiregular. Then either $\Delta_{ii}^+ - \delta_{ii}^+ \geq 2$ or $\Delta_{ii}^- - \delta_{ii}^- \geq 2$. If $\Delta_{ii}^+ - \delta_{ii}^+ \geq 2$, then we can get distinct u, v, w in V_i such that in G_{ii} outdegrees of u, v are $\Delta_{ii}^+, \delta_{ii}^+$ respectively and $uw \in E(G)$, $vw \notin E(G)$. We then obtain G' from G by replacing uw by vw in $E(G)$. Note $S_{G'} = S_G$. Further $f(G') < f(G)$, a contradiction.

If $\Delta_{ii}^- - \delta_{ii}^- \geq 2$, then there exist distinct u, v, w in V_i such that in G_{ii} indegrees of u, v are $\Delta_{ii}^-, \delta_{ii}^-$ respectively and $wu \in E(G)$, $wv \notin E(G)$. Also there is $m \neq i$ suchthat $d_{G_{mi}}^-(v) > d_{G_{mi}}^-(u)$ and there is $x \in V_m$ such that $xu \notin E(G)$, $xv \in E(G)$. $G \to I(wuxv) \to G'$. Then $f(G') < f(G)$, a contradiction. Hence G_i is semiregular.

Now, assuming that for $1 \leq i \leq n$, G_i is semiregular, it can be similarly shown that G_{ij} is semiregular on both sides for $1 \leq i \neq j \leq n$.

Hence the lemma is proved.

We now give the theorem characterizing unidigraphic integer-pair sequences.

Theorem 3.1. *Let S be an integer-pair sequence. Then S is unidigraphic if and only if S is digraphic and satisfies the following conditions:*

P1. *Either $X_i^r = Y_i^r = 0$ or $X_i^r = l_i^r$ or $Y_i^r = l_i^r$ for $1 \leq i \leq n$.*

P2. $k'(d_i, d_i) \in \{0, 1, l_i^r(l_i^r - 1) - 1, l_i^r(l_i^r - 1)\}$ *for* $1 \leq i \leq n$.

P3. *If* $1 \leq i \neq j \leq n$ *and* $l_i^r, l_j^r \geq 2$, *then* $k'(d_i, d_j) \in \{0, 1, l_i^r l_j^r - 1, l_i^r l_j^r\}$.

P4. *If* $k'(d_i, d_i) \notin \{0, l_i'(l_i'-1)\}$, *then* $X_i' = Y_i' = l_i'$ *for* $1 \leqslant i \leqslant n$.
P5. *If* $1 \leqslant i \neq j \leqslant n$, $l_i' \geqslant 2$ *and* $k'(d_i, d_j) \notin \{0, l_i' l_j'\}$, *then for all* $m \neq i, j$, 0 $k'(d_m, d_i)$, $k'(d_i, d_m) = 0$ *or* $l_i' l_m'$; *and* $k'(d_i, d_i) = 0$ *or* $l_i'(l_i'-1)$.
P6. *If* $0 \in A^*$, *then either* $k(0) = 1$ *or there is* j *such that* $l_j' = 1$ *and* $k(0) = k'(0, d_j)$.

Proof. In the following we take G to be the canonical realization of Lemma 3.1. Further if $0 \in A^*$ we assume that $V(G) = V = \bigcup_{i=0}^{n} V_i$ where $|V_0| = k(0)$ and for all $x \in V_0$, $d_G^-(x) = 0$. As G has no isolated vertices so $d_G^+(x) = 1$ for all $x \in V_0$.

We also write $\Delta_{ij}^+, \Delta_{ij}^-, \delta_{ij}^+, \delta_{ij}^-, d_{ij}^+(x), d_{ij}^-(x)$ for $\Delta_{ij}^+(G)$, $\Delta_{ij}^-(G)$, $\delta_{ij}^+(G)$, $\delta_{ij}^-(G)$, $d_{G_{ij}}^+(x)$, $d_{G_{ij}}^-(x)$ respectively.

Now to prove the necessity.

(1) The proof of that P1 holds is similar to that of Assertion 1.

(2) Suppose $2 \leqslant k'(d_i, d_i) \leqslant l_i'(l_i'-1) - 2$. Now as G_{ii} is semiregular so $\Delta_{ii}^+ = \Delta_{ii}^-$ and $\delta_{ii}^+ = \delta_{ii}^-$. So if $\Delta_{ii}^+ = 1$ or $l_i' - 1$, then we have distinct vertices x, y, u, v in V_i such that $xu, yv \in E(G)$ and $xv, yu \notin E(G)$. We then obtain G' from G by replacing xu by yu in $E(G)$ where we assume, without loss of generality, that $d_{ii}^+(y) \geqslant d_{ii}^+(x)$. Then $G'[V_i]$ is not semiregular, contradicting the fact that S is unidigraphic. So we take $2 \leqslant \Delta_{ii}^+ \leqslant l_i' - 2$. Let $x \in V_i$ be such that $d_{ii}^+(x) = \Delta_{ii}^+$. So there is $y \in V_i$ such that $xy \notin V_i$. But $d_{ii}^-(y) \geqslant \delta_{ii}^- \geqslant 1$. Hence there is $w \in V_i$ such that $wy \in E(G)$. So we obtain G' from G by replacing wy by xy in $E(G)$. Then $\Delta_{ii}^+(G') > \Delta_{ii}^+$. A contradiction. Hence P2 holds.

(3) Suppose we have $i \neq j$, $l_i', l_j' \geqslant 2$ and $2 \leqslant k'(d_i, d_j) \leqslant l_i' l_j' - 2$. Then G_{ij} considered as an undirected bipartite graph satisfies the conditions of Lemma 2.2. Hence we get x, y, u, v as stated there. Let $d_{ij}^+(x) \geqslant d_{ij}^+(y)$. We replace yv by xv to get a contradiction as above. Hence P3 holds.

(4) Clearly P4 holds if $l_i' = 2$. So let $l_i' \geqslant 3$. By P2, $k'(d_i, d_i) = 1$ or $l_i'(l_i'-1) - 1$. Hence there is w in V_i which has unique indegree in G_{ii}. If $X_i' \neq l_i'$ or $Y_i' \neq l_i'$ then there are $j, m \neq i$, $0 \leqslant j \neq m \leqslant n$ and vertices $x, y \neq w$ in V_i such that $d_{ji}^-(x) > d_{ji}^-(y)$ and $d_{mi}^-(y) > d_{mi}^-(x)$. So in G_{ii} the only arc or nonarc may be taken to be xw or yw to give two non-isomorphic realizations of S. Hence P4 holds.

(5) Suppose there is $m \neq i, j$ such that $0 < k'(d_m, d_i) < l_i' l_m'$. Then there are $x, y \in V_i$ such that $d_{mi}^-(x) > d_{mi}^-(y)$. Then we may have $d_{ij}^+(x) > d_{ij}^+(y)$ or $d_{ij}^+(y) > d_{ij}^+(x)$ to give two nonisomorphic realizations of S. A contradiction. Similarly it may be shown that $k'(d_i, d_i) = 0$ or $l_i'(l_i'-1)$. Hence P5 holds.

(6) If P6 does not hold, then we can get a realization in which the number of vertices with indegree 0 is less than $k(0)$. A contradiction. Hence P6 holds.

Now to prove the sufficiency.

Let S satisfy the conditions and H be any realization of S. From P2 and P3 we see that $\Delta_{ij}^+(H) = \Delta_{ij}^+$, $\Delta_{ij}^-(H) = \Delta_{ij}^-$, $\delta_{ij}^+(H) = \delta_{ij}^+$ and $\delta_{ij}^-(H) = \delta_{ij}^-$. Also from P6 we see that there are $k(0)$ vertices in $V(H)$ of indegree 0. Hence we can take $V(H) = V(G) = V$.

Let $g_{ij}^+(x)$ and $g_{ij}^-(x)$ denote the outdegree and indegree respectively of x in $H[V_i, V_j]$, where $x \in V_i$ and V_j respectively.

Let $f(x) = ((d_{i1}^+(x), d_{i2}^+(x), \ldots, d_{in}^+(x)), (d_{0i}^-(x), d_{1i}^-(x), \ldots, d_{ni}^-(x)))$ and $f'(x) = ((g_{i1}^+(x), g_{i2}^+(x), \ldots, g_{in}^+(x)), (g_{0i}^-(x), g_{1i}^-(x), \ldots, g_{ni}^-(x)))$ if $x \in V_i$. If $k(0) = 0$, then $d_{0i}^-(x) = g_{0i}^-(x) = 0$ for all $x \in V$.

A map $\phi: V \to V$ is said to be permissible if ϕ is 1–1 and $x \in V_i$, $0 \leq i \leq n$, implies $\phi(x) \in V_i$ and $f'(x) = f(\phi(x))$.

Claim. *If $l_i' \geq 2$, then $\{f(x): x \in V_i\} = \{f'(x): x \in V_i\}$.*

Suppose there is a $j \neq i$ such that $\Delta_{ij}^+ \neq \delta_{ij}^+$, then as S is digraphic so $j \neq 0$. Hence from P5 we get that $\Delta_{ii}^+ = \delta_{ii}^+$, $\Delta_{ii}^- = \delta_{ii}^-$ and for all $m \neq i, j, 0 \Delta_{im}^+ = \delta_{im}^+$, $\Delta_{mi}^- = \delta_{mi}^-$. Also by P6 $\Delta_{0i}^- = \delta_{0i}^- = 0$ as indegree in H of all l_i' vertices of V_i is same. Hence the claim holds in this case.

Suppose $\Delta_{ii}^+ \neq \delta_{ii}^+$ and there is no $j \neq i$, $1 \leq j \leq n$, such that $\Delta_{ij}^+ \neq \delta_{ij}^+$. Note there is a vertex of unique outdegree distinct from the vertex of unique indegree in both G_{ii} and H_{ii} by P2. From P4 we get that $X_i' = Y_i' = l_i'$ and hence the claim holds.

So for all j, $1 \leq j \leq n$, $\Delta_{ij}^+ = \delta_{ij}^+$. If there is a $p, 0 \leq p \leq n$ such that $\Delta_{pi}^- \neq \delta_{pi}^-$, then also Claim holds as we know that either $X_i' = l_i'$ or $Y_i' = l_i'$ by P1. If there is no such p, then, too, the claim holds.

Hence the claim is proved.

It follows from the above claim, as we need not check for V_i with $l_i' = 1$ and for V_0, that there is a permissible map ϕ. It can easily be seen that ϕ is an isomorphism from H onto G.

This completes the proof of the theorem.

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